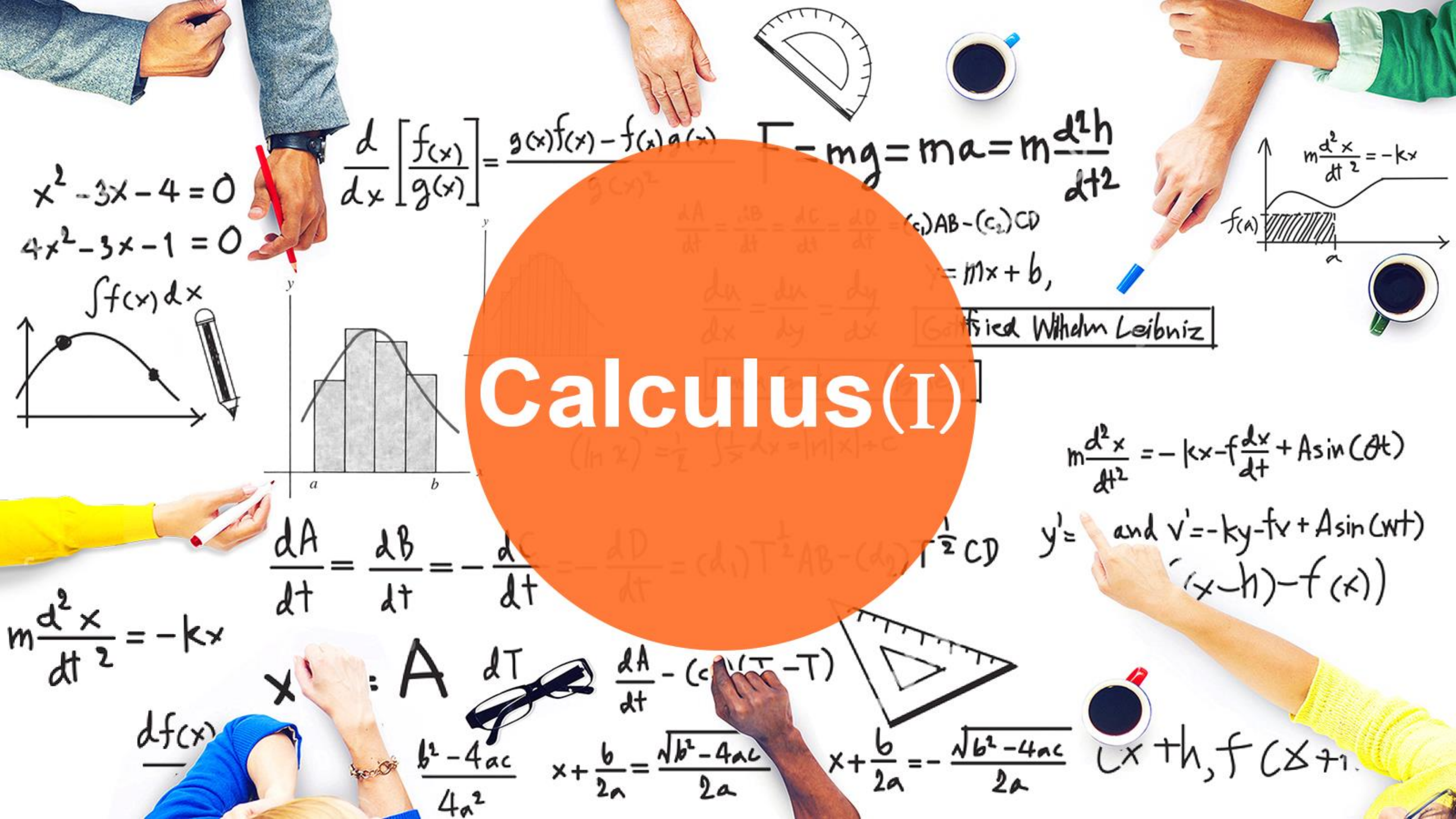
Calculus(I)



4.4 The Second Fundamental Theorem of Calculus and the Method of Substitution

Lecturer: Xue Deng

A distance can be definited as $\int_{T_1}^{T_2} v(t)\mathrm{d}t$ (velocity)

and it can be expressed as $s(T_2)-s(T_1)$ (the difference of distances)

Is there any relationship between $\int_{T_1}^{T_2} v(t)\mathrm{d}t$ and $s(T_2)-s(T_1)$

$\therefore \int_{T_1}^{T_2} v(t)\mathrm{d}t = s(T_2)-s(T_1).$

Find $s'(t)=v(t)$

Theorem A: Second Fundamental Theorem of Calculus

Let $f(x)$ be continuous on $[a, b]$, and let $F(x)$ be any anti-derivative of $f(x)$ on $[a, b]$, $\int_a^b f(x)dx = F(b) - F(a).$

Let $\Phi(x) = \int_a^x f(t)dt,$ so it is an antiderivative of $f(x)$, too.

Both $F(x)$ and $\Phi(x)$ are anti-derivatives of $f(x)$, so there is C making

$$F(x) - \Phi(x) = C, \quad x \in [a,b]$$

1) $x = a$: $F(a) - \Phi(a) = C$ and $\Phi(a) = 0$ so $C = F(a)$, $F(x) - \Phi(x) = F(a).$

2) $x = b$: $F(b) - \Phi(b) = F(a),$ so $\Phi(b) = F(b) - F(a),$ namely, $\int_a^b f(x)dx = F(b) - F(a).$

Theorems

So, we have: $\int_a^b f(x)\mathrm{d}x = F(x)\Big|_a^b = [F(x)]_a^b = F(b) - F(a)$

The problem of finding the integral problem can be translated into

finding the anti-derivative function : $F(x)$

When $a \leq b$, $\int_a^b f(x)dx = F(b) - F(a)$ holds.

When $a > b$, $\int_a^b f(x)dx = F(b) - F(a)$ still holds.

Theorem B: Substitution Rule for Indefinite Integrals

Substitution Rule for Definite Integral

If $f(x) \in C[a,b]$, the function $x = \varphi(t)$ satisfies the following conditions:

(1) $\varphi(\alpha) = a, \varphi(\beta) = b;$

(2) $\varphi(t)$ has a continuous derivative on $[a,b]$ and $R_\varphi \subset [a,b]$.

Then $$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\mathrm{d}t$$

When $\alpha > \beta$, this theorem still holds.

Theorem C: Substitution Rule for Definite Integrals

Let g have a continuous derivative on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

where $u = g(x)$.

Example 1

Show that $\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2}$.

$F(x) = x^2/2$ is an antiderivative of $f(x) = x$.

Therefore,

$$\int_a^b x dx = F(b) - F(a) = \frac{b^2}{2} - \frac{a^2}{2}$$

Example 2

? Evaluate $\int_0^4 \sqrt{x^2+x}(2x+1)dx$.

Let $u = x^2 + x$; then $du = (2x+1)dx$. Thus,

$$\int \underbrace{\sqrt{x^2+x}}_{u}\underbrace{(2x+1)dx}_{du} = \int u^{1/2}du \quad = \frac{2}{3}u^{3/2} + C \quad = \frac{2}{3}(x^2+x)^{3/2} + C$$

Therefore, by the Second Fundamental Theorem of Calculus,

$$\int_0^4 \sqrt{x^2+x}(2x+1)dx \quad = \left[\frac{2}{3}(x^2+x)^{3/2} + C\right]_0^4 \quad = \left[\frac{2}{3}(20)^{3/2} + C\right] - [0 + C]$$

$$= \frac{2}{3}(20)^{3/2} \quad \approx 59.63$$

Example 3

Evaluate $\int_0^{\pi/4} \sin^3 2x \cos 2x \, dx$.

Let $u = \sin 2x$; then $du = 2\cos 2x \, dx$. Thus,

$$\int \sin^3 2x \cos 2x \, dx = \frac{1}{2}\int (\sin 2x)^3 (2\cos 2x)dx = \frac{1}{2}\int u^3 du = \frac{1}{2}\frac{u^4}{4} + C = \frac{\sin^4 2x}{8} + C$$

Therefore, by the Second Fundamental Theorem of Calculus,

$$\int_0^{\pi/4} \sin^3 2x \cos 2x \, dx = \left[\frac{\sin^4 2x}{8}\right]_0^{\pi/4} = \frac{1}{8} - 0 = \frac{1}{8}$$

Example 4

$$\int_a^b f(x)dx = F(b) - F(a).$$

Find $\int_0^{\frac{\pi}{2}} (2\cos x + \sin x - 1)dx.$

The integral $I = \left[2\sin x - \cos x - x\right]_0^{\frac{\pi}{2}}$

$$= 3 - \frac{\pi}{2}.$$

Example 5

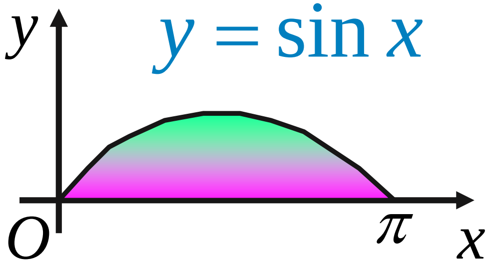
Define A to be the area under the curve of $y = \sin x$, above the x-axis, and between the vertical lines $x = 0$ and $x = \pi$. Evaluate the area A.

$$A = \int_0^{\pi} \sin x \, dx$$

$$= \left[-\cos x\right]\Big|_0^{\pi}$$

$$= -\left[(-1) - 1\right]$$

$$= \mathbf{2.}$$



Example 6

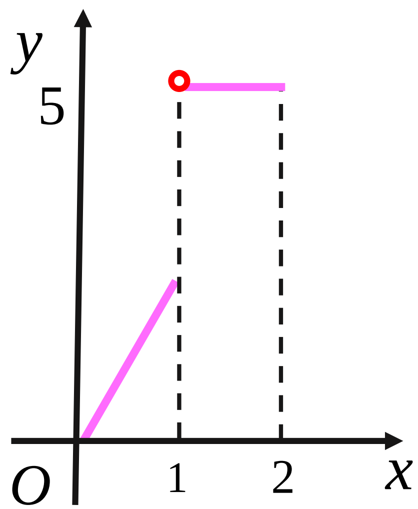
Let $f(x)=\begin{cases}2x, & 0\le x\le 1,\\ 5, & 1<x\le 2,\end{cases}$ evaluate $\int_0^2 f(x)\mathrm{d}x.$

$$\int_0^2 f(x)\mathrm{d}x$$

$$=\int_0^1 f(x)\mathrm{d}x+\int_1^2 f(x)\mathrm{d}x$$

$$=\int_0^1 2x dx+\int_1^2 5dx$$

$$\mathbf{=6.}$$



piecewise function

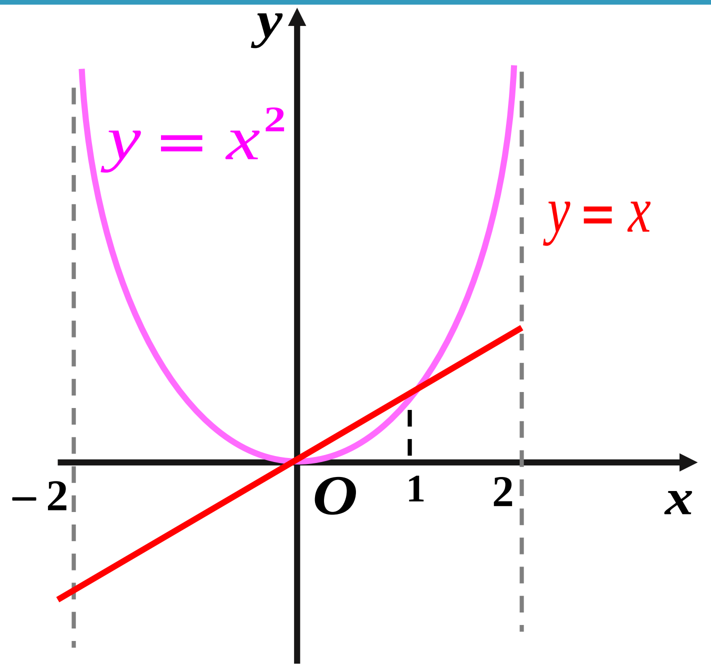
Example 7

Find $\int_{-2}^{2} \max\{x, x^2\}\mathrm{d}x.$

As shown in the figure

$f(x) = \max\{x, x^2\}$

$$= \begin{cases} x^2, & -2 \le x \le 0 \\ x, & 0 \le x \le 1 \\ x^2, & 1 \le x \le 2 \end{cases}$$



$\therefore$ The integral $= \int_{-2}^{0} x^2\mathrm{d}x + \int_{0}^{1} x\mathrm{d}x + \int_{1}^{2} x^2\mathrm{d}x = \dfrac{11}{2}.$

Example 8

? Find $\int_0^{\frac{\pi}{2}} \sin^3 x\mathrm{d}x.$

(Method 1)

$t = \cos x$ $\begin{cases} x = 0, & t = 1 \\ x = \frac{\pi}{2}, & t = 0 \end{cases}$

$$\int_0^{\frac{\pi}{2}} \sin^3 x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x\mathrm{d}x$$

$$= -\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\mathrm{d}\cos x \overset{t = \cos x}{=\!=\!=} -\int_1^0 (1-t^2)\mathrm{d}t$$

$$= -\left(t - \frac{1}{3}t^3\right)\Bigg|_1^0 = \frac{2}{3}.$$

The new variable t , and the upper limit 0 、the lower limit 1 .

Example 8

Find $\int_0^{\frac{\pi}{2}} \sin^3 x\mathrm{d}x.$

(Method 2) $\int_0^{\frac{\pi}{2}} \sin^3 x\mathrm{d}x$

$$=\int_0^{\frac{\pi}{2}} \sin^2 x \sin x\mathrm{d}x$$

$$=-\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\mathrm{d}\cos x$$

$$=-\left[\cos x-\frac{1}{3}\cos^3 x\right]\Bigg|_0^{\frac{\pi}{2}}$$

$$=\frac{2}{3}.$$

Summary

$$\int_a^b f(x)\mathrm{d}x = F(x)\Big|_a^b = \left[F(x)\right]_a^b = F(b) - F(a)$$

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f\left[\phi(t)\right]\phi'(t)\mathrm{d}t \quad \text{(substitution)}$$

Questions and Answers

Find $\displaystyle\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{\mathrm{d}x}{x\sqrt{\ln x(1-\ln x)}}$.

$\mathrm{d}(\sqrt{\ln x}) =$

The integral $\displaystyle = \int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{\mathrm{d}(\ln x)}{\sqrt{\ln x(1-\ln x)}} = \int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{\mathrm{d}(\ln x)}{\sqrt{\ln x}\sqrt{(1-\ln x)}}$

$$= 2\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{\mathrm{d}\sqrt{\ln x}}{\sqrt{1-(\sqrt{\ln x})^2}} = 2\Big[\arcsin(\sqrt{\ln x})\Big]_{\sqrt{e}}^{e^{\frac{3}{4}}}$$

$$= 2\cdot\frac{\pi}{12} = \frac{\pi}{6}.$$

Questions and Answers

Evalute $\int_0^a \sqrt{a^2 - x^2}\,dx\,(a > 0)$.

$$\int_0^a \sqrt{a^2 - x^2}\,dx \xlongequal{\text{let } x = a\sin t} \int_0^{\frac{\pi}{2}} a\cos t \cdot a\cos t\,dt$$

$$= \int_0^{\frac{\pi}{2}} a^2 \cos^2 t\,dt = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)\,dt$$

$$= \frac{a^2}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{1}{4}\pi a^2.$$

Tip: $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 t} = a\cos t,\quad dx = a\cos t\,dt.$

When $x = 0$, so $t = 0$; when $x = a$, so $t = \frac{\pi}{2}$.

Questions and Answers

Find $\int_0^1 |x(2x-1)|\,dx.$

Let $x(2x-1)=0 \Rightarrow x=0, x=\frac{1}{2}.$

when $0 \le x \le \frac{1}{2}$, $x(2x-1) \le 0;$

when $\frac{1}{2} \le x \le 1$, $x(2x-1) \ge 0.$

The integral $= -\int_0^{\frac{1}{2}} x(2x-1)\,dx + \int_{\frac{1}{2}}^1 x(2x-1)\,dx = \frac{1}{4}.$

The Second Fundamental Theorem of Calculus and the Method of Substitution